Announcements

1) New EC up, due in 2 weeks

2) New HW up, due next week

3) Thursday - Lab!

Power Series

section 11.8

Motivation What else?

Differential Equations!

Transverse motion of

a circular membrane

- like an eardrum

You encounter (after some work) the differential equation

$$x\, y''(x) + y'(x) + x\, y(x) = 0$$

How to solve for y?

Only way I know how to solve:

Suppose $y(x)$ is an "infinite polynomial"

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate term by term

$$y'(x) = \sum_{n=0}^{\infty} a_n n \cdot x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2}$$

Plug into equation,
solve for a_n.

Might do this later
if we have time.

Maybe the solution is
somehow a function we
know in disguise

(It isn't)

An "infinite polynomial"

is called a power series

power = "powers of x"

We will be interested in figuring out for what values of x the series converges

Recall:

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}, |x|<1$$

(replaced r with x)

Diverges if $|x| \geq 1$.

How do we tell where a series converges?

Using the ratio test:

Compute

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \left| \frac{a_{n+1}}{a_n} x \right|$$

$$= \left| \frac{a_{n+1}}{a_n} \right| \cdot |x|$$

take the limit!

If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| < 1$, converges

If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| > 1$, diverges

There is always one value of x for which

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges!}$$

$x=0$ makes every term but the first into zero, you get a_0 for the sum.

3 possibilities

1) Series converges only at $x=0$.

2) Series converges for all real numbers.

3) There is a number $R>0$ for which all numbers in $(-R,R)$ make the series converge, and all numbers with $|x|>R$ make the series diverge.

The number R in 3) is called the radius of convergence.

For 1), we say $R = 0$.

For 2), we say $R = \infty$

Example 1: (infinite)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Find radius of convergence.

Ratio Test

$$a_n = \frac{x^n}{n!}$$

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \left|\frac{x^{n+1}}{x^n}\,\frac{n!}{(n+1)!}\right|$$

$$= \left|x\cdot\frac{1}{n+1}\right|$$

$$= \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1}$$

$$= |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

Since this holds for all x, the series converges for all x and $R = \infty$.

Example 2! (zero)

$$\sum_{n=0}^{\infty} n^n x^n$$ - Determine radius of convergence

$$a_{n+1} = (n+1)^{n+1} x^{n+1}$$

$$a_n = n^n x^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right|$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}\ x^{n+1}}{n^n\ x^n}\right|$$

$$= \left|\frac{(n+1)^n \cdot (n+1)}{n^n} \cdot x\right|$$

$$= |x| \left(\frac{n+1}{n}\right)^n \cdot (n+1)$$

If $x=0$, the whole product is $0<1$, so converges.

If $x \neq 0$,

$$\lim_{n\to\infty}\left(|x|\left(\frac{n+1}{n}\right)^n \cdot (n+1)\right) = |x| \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \cdot (n+1|$$

Using $e^{\ln}$ trick,

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e.$$

Since $\lim_{n\to\infty} n+1 = \infty$,

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \cdot (n+1) = \infty.$$

Since $\infty > 1$, the series diverges for all $x \neq 0$.

Then $R = 0$.

Example 3: (interval)

$$\sum_{n=2}^{\infty}\left(\frac{x^n}{n+1}\cdot 4^n\right)$$

Find radius of convergence

Ratio Test

$$a_n = \frac{x^n}{n+1}\cdot 4^n = \frac{(4x)^n}{n+1}$$

$$a_{n+1} = \frac{(4x)^{n+1}}{n+2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(4x)^{n+1}}{n+2} \cdot \frac{n+1}{(4x)^n}\right|$$

$$= \left|\frac{(4x)^{n+1}}{(4x)^n} \cdot \frac{n+1}{n+2}\right|$$

$$= \left|4x \cdot \frac{n+1}{n+2}\right|$$

$$= 4|x| \cdot \frac{n+1}{n+2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(4 \cdot |x| \cdot \frac{n+1}{n+2}\right)$$

$$= 4 \cdot |x| \cdot \lim_{n\to\infty}\frac{n+1}{n+2}$$

$$4 \cdot |x| \cdot \lim_{n \to \infty} \frac{n+1}{n+2}$$

$$= 4 \cdot |x| \cdot 1 = 4|x|.$$

The ratio test says the series converges whenever $4|x| < 1$.

This is $|x| < \frac{1}{4} = R$

Converges for $|x| < \frac{1}{4}$

Diverges for $|x| > \frac{1}{4}$

Ratio test fails for

$|x| = \frac{1}{4}$, which

$x = \pm \frac{1}{4}$.

Plug these numbers back in for

x into the original series

$$\sum_{n=2}^{\infty} \frac{(4x)^n}{n+1}.$$ Evaluate

convergence or divergence

using a test that is NOT

the ratio test.

$x = \frac{1}{4}$

$$\sum_{n=2}^{\infty} \frac{\left(4 \cdot \frac{1}{4}\right)^n}{n+1} = \sum_{n=2}^{\infty} \frac{1^n}{n+1}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n+1}$$

Use integral test. (guess divergent)

$$\int_2^{\infty} \frac{1}{x+1}\,dx \geq \int_2^{\infty} \frac{1}{x+x}\,dx$$

$$= \int_2^{\infty} \frac{1}{2x}\,dx = \frac{1}{2}\int_2^{\infty} \frac{1}{x}\,dx$$

The last integral diverges by p-rule $(p = 1)$, so the series diverges.

This means $x = \frac{1}{4}$ does not produce a convergent series.

$x = -\frac{1}{4}$

$$\sum_{n=2}^{\infty} \frac{\left(4\cdot\left(-\frac{1}{4}\right)\right)^n}{n+1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1}$$

We do not yet know a test that will work!

(test for divergence fails, integral test does not apply)

Use Alternating Series Test

Suppose $(b_n)_{n=1}^{\infty}$ is a sequence with $b_n \geq 0$ for all n

and

1) $(b_n)_{n=1}^{\infty}$ decreases

$(b_{n+1} \leq b_n$ for all $n)$

2) $\lim_{n \to \infty} b_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

So for

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n+1},$$

$b_n = \frac{1}{n+1} \geq 0$, decreases,

$\lim_{n \to \infty} b_n = 0$, so by

Alternating series Test,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n+1}$$ converges!

$x = -\frac{1}{4}$ is good.

We get the

Interval of Convergence

$$\boxed{\left[-\tfrac{1}{4}, \tfrac{1}{4}\right)}$$

(all the numbers for which the series converges)